# Unit-1

* **Subgroup** $\Rightarrow$ Let $(G, *)$ be a group and $H \subset G$, i.e., $H$ is a non-empty subset of $H$, then $H$ is called subgroup of $G$ if $a, b \in H \Rightarrow ab^{-1} \in H$

or

$a, b \in H \Rightarrow a * b \in H$
$a \in H \Rightarrow a^{-1} \in H$.

* **Coset** $\Rightarrow$ Let $G$ be a group and $H$ be a subgroup of $G$, then right coset is defined by

$$Hx = \{hx : h \in H \ \forall \ x \in G\}.$$

and left coset is defined by

$$xH = \{xh : h \in H \ \forall \ x \in G\}.$$

* **Property of coset** $\Rightarrow$

(i). $Hx = Hy \Leftrightarrow xy^{-1} \in H$.

(ii). $H + x = H + y \Leftrightarrow x - y \in H$.

(iii). $Hx = H \Leftrightarrow x \in H$.

(iv). $H \subseteq H.K$ and $K \subseteq H.K$

* **Normal Subgroup** ⇒ Let $(G, *)$ be a group and ~~left~~ let $H$ be a subgroup of $G$, then $H$ be a normal subgroup of $G$ if

$$x \in G, \quad h \in H \Rightarrow xhx^{-1} \in H$$

∵ $x \in G, \quad n \in N \Rightarrow xnx^{-1} \in N$.

$N$ is a normal subgroup of $G$.

or

$$x \in G, \quad h \in H \Rightarrow xH = Hx \quad \forall x \in G.$$

* **Quotient group** ⇒ Let $(G, *)$ be a group and $H$ be a normal subgroup of $G$, the quotient group $\frac{G}{H}$ is defined by

$$\frac{G}{H} = \{ Hx \mid \forall x \in G \}.$$

* **Proper Subgroup** ⇒ A proper subgroup of $G$ is subgroup of $G$ which is other than $\{e\}$ or $G$.

* **Improper Subgroup** ⇒

* Finite group ⇒ If a group $G$ has finite number of elements then $G$ is a finite group.

$G = \{1, \omega, \omega^2\}$, $|G| = o(G) = 3$

$G = \{-1, 1, i, -i\}$, $|G| = o(G) = 4$.

* Infinite group ⇒ If a group $G$ has infinite number of elements, then $G$ is infinite group.

* Simple group ⇒ If a group $G$ is said to be simple if it has no proper subgroup.

* Normal series ⇒ A sequence $\{G_1, G_2, G_3, \ldots, G_n\}$ of normal subgroup of group $G$ is said to be normal series if

$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$

if $\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$.

then sequence $\{G_1, G_2, \ldots, G_n\}$ is said to be normal series of $G$.

Note ⇒

(i). Each $G_{i-1}$ is normal subgroup of $G_i$ for $i = 1, 2, 3, \ldots$

(ii). $\frac{G_i}{G_{i-1}}$ is quotient group if it is also a factors of $G$.

(iii). $\frac{G_1}{G_0}, \frac{G_2}{G_1}, \frac{G_3}{G_2}, \ldots, \frac{G_i}{G_{i-1}} \ldots \frac{G_r}{G_{r-1}} = G$ are all factors of $G$.

* Composition series ⇒ A composition series of a group $G$ is a normal series $(G_0, G_1, \ldots, G_r)$ without repetition whose factors $\frac{G_i}{G_{i-1}}$ are all simple groups. The factors $\frac{G_i}{G_{i-1}}$ are called composition factors of $G$.

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$D_1 = \{1\}$, $D_2 = \{1, 2\}$, $D_3 = \{1, 3\}$, $D_5 = \{1, 5\}$

$D_6 = \{1, 2, 3, 6\}$, $D_{10} = \{1, 2, 5, 10\}$

$D_{15} = \{1, 2, 3, 5, 15\}$, $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

$\{e\} = D_1 \subseteq D_2 \subseteq D_6 \subseteq D_{30}$

$\{e\} = D_1 \subseteq D_3 \subseteq D_6 \subseteq D_{30}$.

$\{e\} = D_1 \subseteq D_2 \subseteq D_{10} \subseteq D_{30}$

$\{e\} = D_1 \subseteq D_5 \subseteq D_{15} \subseteq D_{30}$.

are ~~all~~ composition series of $D_{30}$.

**Theorem - 1.1** Every finite group has a composition series

Proof ⇒ Let $G$ is a finite group and let $|G| = n$, we will prove the theorem by mathematical induction method on $|G|$.

Case-(i). If $n = 1$,

$$|G| = 1$$

i.e. $G = \{e\}$

Here $\{e\} = G_0 \subseteq G_1 = G$ is only composition series of $G$.

Hence the theorem true for $n = 1$.

Case-(ii). Assume that the theorem is true for all group of order less than $n$.

Let $H$ be a proper subgroup of $G$. Then

$$|H| < |G|$$

$$|H| < n.$$

By our induction hypothesis, $H$ has a composition series.

Let $\{e\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r = H$ be a composition series of $H$.

$\because H \subseteq G$.

$\{e\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subset H_r = H \subseteq G$ is a composition series of $G$.

Hence every finite group has a composition series.

Hence Proved

*. Theorem - 1.3 Jordan - Holder theorem $\Rightarrow$ Any two composition series of a finite group are equivalent.

Proof $\Rightarrow$ Let $G$ is finite group.

Case - I If $|G| = 1$ or if $G$ is a simple group, then $G = \{e\}$

Here $\{e\} = G_0 \subseteq G_1 = G$ is only composition series of $G$.

Thus, the theorem is true for $n = 1$.

Case-II. If $o(G) > 1$ or if $G$ is not a simple group. Then we assume that the theorem is true for all group whose order less than $o(G)$.

Consider two composition series of $G$.

$S_1 : \{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{r-1} \subset G_r = G$ — ①

$S_2 : \{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{s-1} \subset H_s = G$ — ②

Now if $G_{r-1} = H_{s-1}$,
then by our induction ~~method~~ hypothesis $S_1$ and $S_2$ are equivalent.

If $G_{r-1} \neq H_{s-1}$.

let $K = G_{r-1} \cap H_{s-1}$

then $K \subseteq G_{r-1}$ and $K \subseteq H_{s-1}$.

$\because$ $G$ is finite group and $K \subseteq G_{r-1} \subset G$.

$\therefore$ $K$ is also a finite group.

$\because$ Every finite group has a composition series.

$\therefore$ $K$ has a composition series.

Let $\{e\} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K$ be composition series of $K$.

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$\because K \subset G_{r-1} \subset G$ and $K \subset H_{s-1} \subset G$.

there are two more composition series of G.

$S_3 : \{e\} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K \subset G_{r-1} \subset G$ — (3)

$S_4 : \{e\} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K \subset H_{s-1} \subset G$ — (4).

By our induction hypotheses, $S_1$ and $S_3$ are equivale[nt]
Similarly $S_2$ and $S_4$ are equivalent.

$\because$ $G_{r-1}$ and $H_{s-1}$ are subgroup of G.
$\therefore$ $G_{r-1}H_{s-1}$ is also subgroup of G.

$H \subset HK$
$K \subset HK$.

And $G_{r-1} \subset G_{r-1}H_{s-1}$
$H_{s-1} \subset G_{r-1}H_{s-1}$.

But $G_{r-1}H_{s-1}$ are maximal subgroup.
$\therefore$ $G_{r-1}H_{s-1} = G$.

* By diamond isomorphism theorem :-

If H and N are two subgroup of G, then

$$\frac{H \cdot N}{N} \cong \frac{H}{H \cap N}$$

$\therefore$ We have, $\dfrac{G_{r-1} \cdot H_{s-1}}{H_{s-1}} \cong \dfrac{G_{r-1}}{G_{r-1} \cap H_{s-1}}$

and $\dfrac{H_{s-1} \cdot G_{r-1}}{G_{r-1}} \cong \dfrac{H_{s-1}}{H_{s-1} \cap G_{r-1}}$

$\cancel{\dfrac{G_1}{G_{r-1}}}$ $\dfrac{G_1}{H_{s-1}} \cong \dfrac{G_{r-1}}{K}$ and $\dfrac{G}{G_{r-1}} \cong \dfrac{H_{s-1}}{K}$.

from eq$^n$ ③ & ④,

$\therefore$ $S_3$ and $S_4$ are equivalent.

~~$\therefore$ $S_1$ and $S_3$~~

$\because$ $S_1 \sim S_3$ and $S_2 \sim S_4$ and ~~$S_3$ and~~ $S_3 \sim S_4$.

$\therefore$ $S_1 \sim S_2$.

Hence two composition series of finite group are equivalent.

* Equivalent $\Rightarrow$ Two normal series $S$ and $S'$ of $G$ is said to be equivalent written $S \sim S'$, if the no. of elements are equal ~~and~~ or factors group of first series is isomorphic to factors group of second series.

* Commutator element $\Rightarrow$ If $a, b \in G$, then $aba^{-1}b^{-1}$ is called commutator element of $G$.

$U$ is a collection of commutator elements of $G$:

i.e., $U = \{ aba^{-1}b^{-1} \mid a, b \in G \}$

Theorem - Let $G'$ be a commutator subgroup of a group $G$, then $G$ is abelian iff $G' = \{e\}$

Proof $\Rightarrow$ Necessary Condition $\Rightarrow$

Suppose that $G$ is a abelian group, then $ab = b.a \quad \forall\ a, b \in G$.

$$(ab)(ba)^{-1} = \{e\}$$

$$a.b.a^{-1}.b^{-1} = \{e\}$$

$$G' = \{e\} \quad \forall\ a, b \in G.$$

Conversely $\Rightarrow$ Suppose that $G' = \{e\}$.

~~Let~~ Then we have to prove that $G$ is abelian

Let $a, b \in G$.

Then, $aba^{-1}b^{-1} = \{e\}$

$(ab)(ba)^{-1} = \{e\}$ (From Reversal law)

$(ab)(ba)^{-1}(ba) = \{e\}(ba)$.

$(ab) \cdot e = ba$

$ab = ba$.

i.e. Commutative property satisfied.

Hence $G$ is abelian. Proved

Theorem- Let $G$ be a group and $G'$ be commutator subgroup of $G$, then

(i). $G'$ is normal in $G$.

(ii). $\frac{G}{G'}$ is abelian.

(iii). If $N$ is any normal subgroup of $G'$, then $\frac{G}{N}$ is abelian iff $G' \subseteq N$.

Proof $\Rightarrow$ (i). Let $g \in G$ and $x \in G'$ $\because G' \subset G$

then $g \in G$ and $x \in G$

$\Rightarrow gxg^{-1}x^{-1} \in G'$

$\because gxg^{-1}x^{-1} \in G'$ and $x \in G'$.

$(gxg^{-1}x^{-1})\,x \in G'$

$\Rightarrow gxg^{-1}e \in G'$

$\Rightarrow gxg^{-1} \in G'$.

$\because g \in G$ and $x \in G'$

$\Rightarrow gxg^{-1} \in G'$

Hence $G'$ is normal subgroup of $G$.

(ii). $\frac{G}{G'}$ is abelian $\Rightarrow$

$$\frac{G}{G'} = \{G'x \mid x \in G\}$$

Let $G'x$ and $G'y \in \frac{G}{G'}$ where $x, y \in G$.

$x, y \in G$

$xyx^{-1}y^{-1} \in G'$. $\quad \because (ab)^{-1} = b^{-1}a^{-1}$.

$(xy)(yx)^{-1} \in G'$ $\quad \because ab^{-1} \in H \Leftrightarrow Ha = Hb$

$G'(xy) = G'(yx)$

$(G'x).(G'y) = (G'y)(G'x)$ $\quad \because H(ab) = H(a).H(b)$.

Hence $\frac{G}{G'}$ is abelian.

(iii). Let $N$ be a normal subgroup of $G$, then $g \in G$, $x \in N \Rightarrow gxg^{-1} \in N$.

First suppose that $\frac{G}{N}$ is abelian.

$$\frac{G}{N} = \{ Nx \mid x \in G \}$$

Let $Nx, Ny \in \frac{G}{N}$, where $x, y \in G$.

$\because$ $\frac{G}{N}$ is abelian.

$\therefore$ $(Nx)(Ny) = (Ny)(Nx)$.

$\Rightarrow$ $N(xy) = N(yx)$.

$\Rightarrow$ $(xy)(yx)^{-1} \in N$

$\Rightarrow$ $xyx^{-1}y^{-1} \in N$

$\Rightarrow$ $G' \subseteq N$

Conversely, Suppose that $G' \subseteq N$.

To prove that $\frac{G}{N}$ is abelian

Let $Nx, Ny \in \frac{G}{N}$ where $x, y \in G$.

$\because$ $x, y \in G$

$\Rightarrow$ $xyx^{-1}y^{-1} \in G'$. $\because$ $G' \subseteq N$.

$xyx^{-1}y^{-1} \in N$. $\because$ Reversal law.

$\Rightarrow xy\,(yx)^{-1} \in N$

$\Rightarrow N(xy) = N(yx)$ $\because ab^{-1} \in H \Leftrightarrow Ha = Hb$.

$\Rightarrow (Nx)(Ny) = (Ny).(Nx)$ $\because H(ab) = H(a).H(b)$

Hence $\frac{G}{N}$ is abelian.

** Solvable group $\Rightarrow$ A group G is said to be solvable group if it has a ~~competition~~ composition series with abelian factor.

or

A group G is solvable iff $G^{(k)} = \{e\}$ for some integer k.

Theorem - A group G is solvable iff $G^{(k)} = \{e\}$ for some integer k.

Proof $\Rightarrow$ Let $G^{(k)} = \{e\}$ for some integer k.

Then we have to show that G is solvable.

$N_0 = G,\ N_1 = G',\ N_2 = G'',\ N_3 = G''', \ldots, N_k = G^{(k)} = \{e\}$.

$$G \supset G' \supset G'' \supset G''' \supset \cdots\cdots \supset G^k = \{e\}.$$

$$N_0 \supset N_1 \supset N_2 \supset N_3 \supset \cdots\cdots \supset N_k.$$

$$G = N_0 \supset N_1 \supset N_2 \supset \cdots\cdots \supset N_k = G^{(k)} = \{e\}.$$

we have.

(i). $N_i$ is normal subgroup of $N_{i-1}$.
(ii). $\frac{N_{i-1}}{N_i}$ is abelian.

Verification $\Rightarrow$

(i). Let $g \in N_0$ and $x \in N_1$

$g \in G$ and $x \in G'$, $G' \subset G$.

$g \in G$ and $x \in G$

$\Rightarrow$ $gxg^{-1}x^{-1} \in G'$ and $x \in G'$

$\Rightarrow$ $(gxg^{-1}x^{-1})x \in G'$

$\Rightarrow$ $(gxg^{-1})(x^{-1}x) \in G'$

$\Rightarrow$ $gxg^{-1} \in G'$

$\Rightarrow$ $gxg^{-1} \in N_1$

$g \in N_0$ and $x \in N_1$ $\Rightarrow$ $gxg^{-1} \in N_1$

$\therefore$ $N_1$ is normal subgroup of $N_0$.

Similarly, $N_2$ is normal subgroup of $N_1$.
$N_3$ is normal subgroup of $N_2$.
$\vdots$

$N_i$ is normal subgroup of $N_{i-1}$.

(ii). $\frac{N_{i-1}}{N_i}$ is abelian.

$\frac{N_0}{N_1}$ is abelian.

$$\frac{N_0}{N_1} = \{N_1 x \mid x \in N_0\}.$$

Let $N_1 x$ and $N_1 y \in \frac{N_0}{N_1}$ $\forall$ $x, y \in N_0$.

$x, y \in N_0 \Rightarrow x, y \in G$
$\Rightarrow xyx^{-1}y^{-1} \in G'$
$\Rightarrow xyx^{-1}y^{-1} \in N_1$

$\Rightarrow (xy)(yx)^{-1} \in N_1$ $\quad \because ab^{-1} \in H \Leftrightarrow Ha = Hb$.

$\Rightarrow N_1(xy) = N_1(yx)$.

$\Rightarrow (N_1 x)(N_1 y) = (N_1 y)(N_1 x)$.

$\therefore \frac{N_0}{N_1}$ is abelian.

Similarly, $\frac{N_1}{N_2}$ is abelian.

$\frac{N_2}{N_3}$ is abelian ......

$\frac{N_{i-1}}{N_i}$ is abelian.

$\therefore$ G is solvable.

II$^{nd}$ Part $\Rightarrow$ Suppose that G is solvable group, then we have to prove that $G^{(k)} = \{e\}$.

Let $G = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \{e\}$ be a normal series of G with $\frac{N_{i-1}}{N_i}$ is abelian.

Now, let $N_i a, N_i b \in \frac{N_{i-1}}{N_i}$, where $a, b \in N_{i-1}$.

$\because \frac{N_{i-1}}{N_i}$ is abelian

$$(N_i a)(N_i b) = (N_i b)(N_i a)$$
$$N_i(ab) = N_i(ba)$$
$$ab(ba)^{-1} \in N_i$$
$$aba^{-1}b^{-1} \in N_i$$

$\therefore N_{i-1}' \subseteq N_i$.

$$N_i \supset N_{i-1}'$$

$i=1$,

$$N_1 \supset N_0' = G^{(1)}.$$

$i=2$,

$$N_2 \supset N_1' \supset (N_0')' = G^{(2)}$$

$i=3$,

$$N_3 \supset N_2' \supset (N_1')' \supset (N_0'')' = G^{(3)}.$$

$i=4$,

$$N_4 \supset N_3' \supset N_2'' \supset N_1''' \supset N_0'''' = G^{(4)}.$$

$$N_k \supset G^{(k)}.$$

$$G^{(k)} \subset N_k = \{e\}.$$

$$G^{(k)} \subset \{e\}.$$

But

$$\{e\} \subset G^{(k)}.$$

$\therefore$ $G^{(k)} = \{e\}$ for some integer $k$.

Theorem – Let $G$ be a group. If $G$ is solvable then every subgroup of $G$ and every homomorphism image of $G$ are solvable. Conversely if $N$ is normal subgroup of $G$ such that $N$ and $\frac{G}{N}$ are solvable then $G$ is solvable.

Proof $\Rightarrow$ I part Let $G$ be a solvable group.
then $G^{(k)} = \{e\}$ for some integer $k$.

Case-I Let $H$ be a subgroup of $G$,
then $H \subset G$.

$$H^{(n)} \subset G^{(n)}. \text{ for some integer } n.$$

i.e. $H^{(k)} \subset G^{(k)} = \{e\}$.

$$H^{(k)} \subseteq \{e\}$$

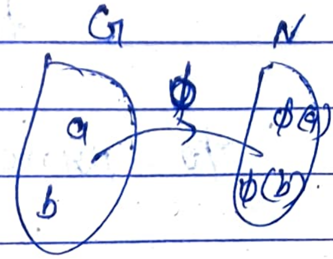
But

$$\{e\} \subset H^{(k)}.$$

Hence $H^{(k)} = \{e\}$

Hence $H$ is solvable group and so every subgroup of $G$ is solvable.

Case-II Let $\phi : G \to N$ be a homomorphism.
then $N$ is homomorphic image of $G$.




Let $a, b \in G$, then
$aba^{-1}b^{-1} \in G'$ and $\phi(a), \phi(b) \in N$.

$$\phi(a)\phi(b)\,\phi(a)^{-1}(\phi(b))^{-1} \in N'$$
$$\phi(a)\cdot\phi(b)\cdot\phi(a^{-1})\cdot\phi(b^{-1}) \in N'.$$

$$\Rightarrow \phi(aba^{-1}b^{-1}) \in N'$$

$$\Rightarrow \phi(G') \subseteq N' \quad — ①$$

But $N' \subset \phi(G') \quad — ②$

$$\therefore \phi(G') = N'.$$

By Induction method,

$$\phi(G^{(k)}) = N^{(k)}.$$
$$\phi(e) = N^{(k)}$$
$$e' = N^{(k)}$$
$$N^{(k)} = \{e\}$$

$\therefore$ N is solvable group and so every homomorphic image of G is solvable group.

II Part $\Rightarrow$ Let N be a normal subgroup of G such that N and $\frac{G}{N}$ are solvable.

To prove that G is solvable, then $N^{(k)} = \{e\}$ and

$$\left(\frac{G}{N}\right)^{l} = \{e\}$$

for some integer k & l.

Formula $\Rightarrow$ $\left(\frac{G}{N}\right)^{n} = \frac{G^{(n)}N}{N}$.

$$\left(\frac{G}{N}\right)^{l} = \left(\frac{G^{(l)}N}{N}\right)$$

$$\{e\} = \frac{G^{(l)}N}{N}$$

$$G^{(l)}N = N\{e\}$$

$$G^{(l)}.N = N$$

$$G^{(l)} \subseteq N$$

$$G^{(l+k)} \subseteq N^{(k)}$$

$$G^{(l+k)} \subseteq \{e\} \quad \text{---} (3)$$

But $\{e\} \subseteq G^{(l+k)}$ --- (4)

from eq$^n$ (3) & (4),

$$\therefore G^{(l+k)} = \{e\}$$

$\because l+k = m$

$$G^{(m)} = \{e\}$$

Hence $G$ is a solvable group.

Theorem – A group $G$ is solvable iff $G$ has a normal series with abelian factor. Further a finite group is solvable iff its composition factor are cyclic group of prime order.

Proof ⇒ First suppose that $G$ is solvable group, then
I part $G^{(k)} = \{e\}$ for some integer $k$.

Let $G = G_0 \supset G' \supset G'' \supset G''' \supset \cdots \supset G^{(k)} = \{e\}$.

Then, $G^{(i)}$ is a normal subgroup of $G^{(i-1)}$ and $\frac{G^{(i-1)}}{G^{(i)}}$ is abelian.

Reverse, Thus $G$ has normal series with abelian factors (conversely) suppose that $G$ has a normal series with abelian factors so

Let $\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{r-1} \subset H_r = G$

be a normal series of $G$ with $\frac{H_i}{H_{i-1}}$ is abelian and $H_{i-1}$ is normal subgroup of $H_i$.

Let $a, b \in H_i$.

Then $\frac{H_i}{H_{i-1}} = \{H_{i-1}a \mid a \in H_i\}$

$H_{i-1}a, H_{i-1}b \in \frac{H_i}{H_{i-1}}$

$\because$ $\frac{H_i}{H_{i-1}}$ is abelian.

$\therefore$ $(H_{i-1}a)(H_{i-1}b) = (H_{i-1}b)(H_{i-1}a)$.

$H_{i-1}(ab) = H_{i-1}(ba)$ $\quad \because (Ha)(Hb) = H(ab)$

$\because Ha = Hb \Rightarrow ab^{-1} \in H$

$(ab)(ba)^{-1} \in H_{i-1}$.

$aba^{-1}b^{-1} \in H_{i-1}$

$H_i' \subseteq H_{i-1}$. Taking $i = r$

$H_r' \subseteq H_{r-1}$.

$G' \subseteq H_{r-1}$.

$G'' \subseteq H_{r-2}$.

$G''' \subseteq H_{r-3}$.

$G^{(i)} \subseteq H_{r-i}$.

$G^{(r)} \subseteq H_{r-r}$

$G^{(r)} \subseteq H_0$

$G^r \subseteq \{e\}$

But, $\{e\} \subseteq G^r$

$\therefore \quad G^r = \{e\}$ for some integer $r$.

Hence $G$ is ~~Abelian~~ Solvable.

Second Part $\Rightarrow$ Suppose that $G$ has finite solvable group then $G$ has a normal series.

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_r = G.$$

$H_{i-1}$ is a normal subgroup of $H_i$ each factor

$\frac{H_i}{H_{i-1}}$ is abelian.

$\because$ $\frac{H_i}{H_{i-1}}$ is a finite abelian group.

Then we get a composition series of $G$ in which every factor is cyclic group of prime order.

**Result:-** Every finite group has a composition series.

Every composition series of abelian group has cyclic group of prime order.

* Center of a group $\Rightarrow$ The set of all element in $G$ that commute every element in $G$ is called center of group $G$, which is written as

$$Z(G) = \{x \mid xy = yx \ \forall \ y \in G\}.$$

Note -(i). $Z_0(G) = \{e\} \quad \forall \ e \in Z_0(G)$.

(ii). $Z_1(G) = \{x \mid xyx^{-1}y^{-1} = e, \ \forall \ y \in G\}$

$$xyx^{-1}y^{-1} \in Z_0(G).$$

(iii). $Z_2(G) = \{x \mid xyx^{-1}y^{-1} \in Z_1(G), \ \forall \ y \in G\}$.

(iv). $Z_3(G) = \{x \mid xyx^{-1}y^{-1} \in Z_2(G),\ \forall y \in G\}$

⋮

(v). $Z_m(G) = \{x \mid xyx^{-1}y^{-1} \in Z_{m-1}(G),\ \forall y \in G\}$

$Z_n(G) = \{x \mid xyx^{-1}y^{-1} \in Z_{n-1}(G),\ \forall y \in G\}$

* Nilpotent group ⇒ A group $G$ is said to be nilpotent if $Z_m(G) = G$ for some integer $m$.

$\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq Z_3(G) \subseteq \cdots \subseteq Z_m(G) = G$

$Z_n(G) = n^{th}$ center of $G$.

$Z\left(\frac{G}{Z_1(G)}\right) =$ center of $\frac{G}{Z_1(G)} = \frac{Z_1(G)}{Z_0(G)}$.

$Z\left(\frac{G}{Z_2(G)}\right) = \frac{Z_2(G)}{Z_1(G)}$.

$Z\left(\frac{G}{Z_3(G)}\right) = \frac{Z_3(G)}{Z_2(G)}$.

$Z\left(\frac{G}{Z_i(G)}\right) = \frac{Z_i(G)}{Z_{i-1}(G)}$.

the smallest $m$ such that $Z_m(G) = G$ is called the class of nilpotency of $G$.

If $G$ is abelian then $Z_1(G) = Z(G) = G$

Theorem– Prove that a group order $p^n$ ($p$ is prime) is nilpotent.

Proof ⇒ Let $o(G) = p^n$.

and let $Z_1(G)$ be the centre of group $G$.
let $o(Z_1(G)) = p^r$.

Now, $o\left(\frac{G}{Z_1(G)}\right) = \frac{o(G)}{o(Z_1(G))} = \frac{p^n}{p^r}$
$= p^{n-r}$
$= p^{n_1}$ where, $n_1 = n-r < n$.

$$o\left(\frac{G}{Z_1(G)}\right) = p^{n_1}$$

Let $Z\left(\frac{G}{Z_1(G)}\right)$ be the centre of $\left(\frac{G}{Z_1(G)}\right)$.

then $Z\left(\frac{G}{Z_1(G)}\right)$ is a subgroup of $\left(\frac{G}{Z_1(G)}\right)$.

So let $o\left(Z\left(\frac{G}{Z_1(G)}\right)\right) = p^{r_1}$, where $1 \le r_1 \le n_1$

then $o\left(\frac{Z_2(G)}{Z_1(G)}\right) = p^{r_1}$.

$\Rightarrow \frac{o(Z_2(G))}{o(Z_1(G))} = p^{r_1}$.

$\Rightarrow o(Z_2(G)) = p^{r_1} \cdot o(Z_1(G))$

$\Rightarrow o(Z_2(G)) = p^{r_1} \cdot p^{r}$

$\Rightarrow o(Z_2(G)) = p^{r_1+r}$

$\because r \leq r_1 + r \leq n_1 + r = n$

$r \leq r_1 + r \leq n$.

$p^{r} \leq p^{r_1+r} \leq p^{n}$.

$\Rightarrow o(Z_1(G)) \leq o(Z_2(G)) \leq o(G)$.

By repeating,

$o(Z_1(G)) \leq o(Z_2(G)) \leq o(Z_3(G)) \leq o(Z_4(G)) \leq \cdots \leq o(Z_n(G)) = o(G)$.

then $o(Z_n(G)) = o(G)$.

$\therefore \boxed{Z_n(G) = G}$ for some integer $n$.

Hence $G$ is nilpotent group.

Theorem – Prove that every nilpotent group is solva[ble]

Proof ⇒ Let $G$ be nilpotent group.

Let $G_0 = Z_0(G) = \{e\}$.

$G_1 = Z_1(G)$.

$G_2 = Z_2(G)$.

$\vdots$

$G_i = Z_i(G)$.

$\vdots$

$G_n = Z_n(G)$.

$\because \{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \cdots \subseteq Z_n(G) = G.$

is a normal series of $G$.

$\therefore \{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$ is a normal series of $G$.

To show that $\frac{G_i}{G_{i-1}}$ is abelian.

Since $Z\left(\frac{G}{Z_{i-1}(G)}\right) = \frac{Z_i(G)}{Z_{i-1}(G)} = \frac{G_i}{G_{i-1}}$

and $Z\left(\frac{G}{Z_{i-1}(G)}\right)$ is center of $\frac{G}{Z_{i-1}(G)}$.

$\therefore$ $Z\left(\frac{G}{Z_{i-1}(G)}\right)$ is abelian.

and so $\frac{G_i}{G_{i-1}}$'s is abelian.

Thus $G$ has a normal series with abelian factor.

Hence $G$ is a solvable group.

Theorem - Prove that every subgroup of nilpotent group is nilpotent.

Proof $\Rightarrow$ Let $G$ be nilpotent group and $H$ be a subgroup of $G$.

Then $Z_n(G) = G$ for some integer $n$.

Let $\alpha \in H \cap Z_1(G)$ then.

$\alpha \in H \cap Z_1(G) \Rightarrow \alpha \in H$ and $\alpha \in Z_1(G)$.

$\Rightarrow \alpha \in H$ and $\alpha x = x\alpha, \forall x \in G$.

$\Rightarrow \alpha \in H$ and $\alpha x = x\alpha \quad \forall x \in H$.

$\Rightarrow \alpha \in Z_1(H)$.

$\alpha \in H \cap Z_1(G) \Rightarrow \alpha \in Z_1(H)$.

$H \cap Z_1(G) \subseteq Z_1(H)$.

Similarly, $H \cap Z_2(G) \subseteq Z_2(H)$.

$H \cap Z_3(G) \subseteq Z_3(H)$.

$\vdots$

$H \cap Z_n(G) \subseteq Z_n(H)$.

$H \cap G \subseteq Z_n(H)$.

$H \subseteq Z_n(H)$. — ①

But $Z_n(H) \subseteq H$ — ②

from eqn ① & ②,

$Z_n(H) = H$

$\therefore$ H is nilpotent.

Every subgroup of nilpotent group is nilpotent.

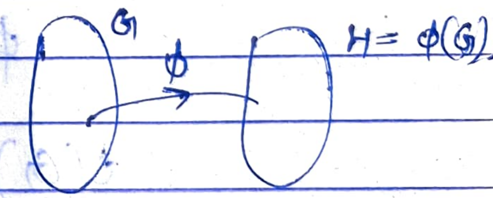
Proved

Theorem-2. Let $G$ be nilpotent group, then prove that every subgroup of $G$ and every Homomorphic image of $G$ are nilpotent.

Proof ⇒ Let $G$ be a nilpotent group.
To prove that every subgroup of nilpotent is nilpotent (same as ~~previous~~ first theorem)

To prove that every homomorphic image of $G$ is nilpotent.

Let $\phi : G \to H$ be a homomorphism. So that $H$ is a homomorphic image of $G$.




Let $x \in Z_1(G)$, then

$x \in Z_1(G) \Rightarrow xy = yx \;\forall\; y \in G.$

$\Rightarrow xy(yx)^{-1} = \{e\}$

$x \in Z_1(G) \Rightarrow xyx^{-1}y^{-1} = \{e\} \;\forall\; y \in G.$

$\phi(x) \in \phi(Z_1(G)) \Rightarrow \phi(xyx^{-1}y^{-1}) = \phi(e)$

$\Rightarrow \phi(x)\,\phi(y)\,\phi(x^{-1})\,\phi(y^{-1}) = e'$

$\cancel{\Rightarrow \phi(x).\phi(y) = \phi y}$

$$\Rightarrow \phi(x)\phi(y)[\phi(x)]^{-1}[\phi(y)]^{-1} = e'$$

$$\Rightarrow \phi(x)\phi(y) = \phi(y)\phi(x) \quad \forall\ \phi(y) \in H.$$

$$\Rightarrow \phi(x) \in Z_1(H)$$

Thus, $\phi(Z_1(G)) \subseteq Z_1(H)$.

Similarly, $\phi(Z_2(G)) \subseteq Z_2(H)$.

$$\phi(Z_3(G)) \subseteq Z_3(H).$$

$$\vdots$$

$$\phi(Z_n(G)) \subseteq Z_n(H) \quad \text{--- (1)}$$

$\phi(G) \subseteq Z_n(H)$ $\because$ G is nilpotent
$Z_n(G) = G$

$H \subseteq Z_n(H)$ --- (2)

$\because \phi(G) = H.$

~~And~~ But we know that

$$Z_n(H) \subseteq H \quad \text{--- (3)}$$

from eqn (2) & (3),

$Z_n(H) = H$ $\forall$ some integer $n$.

$\therefore$ H is nilpotent.

Hence every homomorphic image of $G$ is nilpotent.

Theorem – A group $G$ is nilpotent iff $G$ has normal series
$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$, such that
$\frac{G_i}{G_{i-1}} \subseteq Z\left(\frac{G}{G_{i-1}}\right)$, $\forall\ i = 1, 2, \cdots, m$.

Proof $\Rightarrow$ Let $G$ be a nilpotent group of class $m$.
$Z_m(G) = G$.

$Z_1(G) = \{x \mid xyx^{-1}y^{-1} = e,\ \forall\ y \in G\}$

$Z_2(G) = \{x \mid xyx^{-1}y^{-1} \in Z_1(G)\ \forall\ y \in G\}$

$Z_3(G) = \{x \mid xyx^{-1}y^{-1} \in Z_2(G)\ \forall\ y \in G\}$

$Z_n(G) = \{x \mid xyx^{-1}y^{-1} \in Z_{n-1}(G)\ \forall\ y \in G\}$.

Taking, $Z_0(G) = G_0 = \{e\}$

$Z_1(G) = G_1$
$Z_2(G) = G_2$
$\vdots$
$Z_m(G) = G_m$.

$\therefore \{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \cdots \subseteq Z_m(G) = G_m$

is normal series of $G$

$\therefore \{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$
normal series of $G$.

And $\dfrac{Z_i(G)}{Z_{i-1}(G)} \subseteq Z\left(\dfrac{G}{Z_{i-1}(G)}\right)$

$$\Rightarrow \frac{G_i}{G_{i-1}} \subseteq Z\left(\frac{G}{G_{i-1}}\right)$$

Proved

Conversely, Suppose that $G$ has a normal series

$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$

Such that $\dfrac{G_i}{G_{i-1}} \subseteq Z\left(\dfrac{G}{G_{i-1}}\right)$

We have to prove that $G$ is nilpotent group

$$\frac{G_i}{G_{i-1}} \subseteq Z\left(\frac{G}{G_{i-1}}\right).$$

Taking $i=1$,

$\dfrac{G_1}{G_0} \subseteq Z\left(\dfrac{G}{G_0}\right) \quad \because Z_0(G) = G_0 = \{e\}$

$$\frac{G_1}{\{e\}} \subseteq \frac{Z_1(G)}{Z_0(G)}$$

$$\frac{G_1}{\{e\}} \subseteq \frac{Z_1(G)}{\{e\}}$$

$$G_1 \subseteq Z_1(G)$$

Similarly,

$$G_2 \subseteq Z_2(G)$$
$$G_3 \subseteq Z_3(G)$$
$$\vdots$$
$$G_m \subseteq Z_m(G).$$

$$G \subseteq Z_m(G) \quad \text{--- (1)}$$

But $\quad Z_m(G) \subseteq G \quad$ --- (2)

from eqn (1) & (2),

$$Z_m(G) = G.$$

Proved

Hence $G$ is a nilpotent group.

Theorem - An abelian group $G$ has a composition series iff $G$ is finite.

Proof $\Rightarrow$ Let $G$ be a abelian group with a composition series.

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_r = G$$

We have to prove that $G$ is finite.

Since $G$ is an abelian group.
$\therefore$ the factor $\frac{G_i}{G_{i-1}}$ is also abelian and simple group and it must be cyclic group of prime order $p_i$ where $i = 1, 2, 3, \ldots, r$.

i.e. $O\left(\frac{G_i}{G_{i-1}}\right) = p_i$ where $p$ is prime.

Formula $\Rightarrow$ $O(G) = \prod_{i=1}^{r} O\left(\frac{G_i}{G_{i-1}}\right)$

$\therefore$ $\prod_{i=1}^{r} p_i = p_1 \cdot p_2 \cdot p_3 \cdots p_r$

$=$ a finite number.

Hence $G$ is finite group.

Conversely, suppose that $G$ is a finite abelian group. Then $G$ has a composition series.

∵ Every finite group has composition series.

Theorem – The Jordan Holder theorem implies the fundamental theorem of arithmetic.

Proof ⇒ Let $G$ be a cyclic group of order $n$. Suppose $n$ has two factorization of positive primes.

Say

$$n = p_1 \cdot p_2 \cdot \cdots \cdot p_r.$$

and

$$n = q_1 \cdot q_2 \cdots q_r.$$

then $G$ has a composition series whose factors are cyclic group of prime orders $p_1, p_2, p_3, \cdots, p_r$ respectively and also a composition series whose factors are cyclic group of prime order $q_1, q_2, \cdots, q_s$ respectively.

By the Jordan – holder theorem the two composition series must be equivalent.

Then

$r = s$ and $p_i = q_i$. Proved